

ON THE GELFAND-KIRILLOV DIMENSION  
OF WEAKLY LOCALLY FINITE DIVISION RINGS

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ABSTRACT. Weakly locally finite division rings were considered in [8]. In this paper, firstly, we prove that weakly locally finite division rings are exactly locally PI division rings, and then we consider the Gelfand-Kirillov dimension of such division rings. It is shown that the GKdim of any weakly locally finite division ring is either non-negative integer or infinite. Moreover, for any integer  $n \geq 0$  or  $n = \infty$ , we construct a weakly locally finite division ring whose GKdim is  $n$ . Further, we investigate some questions related with the Kurosh Problem for division rings. In particular, we give a positive answer to the Kurosh Problem for weakly locally finite division rings. Finally, one of Herstein's conjectures is also investigated.

1. INTRODUCTION

In the theory of algebras, it is well-known that an algebra is locally finite iff its Gelfand-Kirillov dimension (GKdim for short) is 0 [15]. In this paper, we are interested in division rings regarding as algebras over their centers and we investigate the GKdim of such rings. Recall that the class of weakly locally finite division rings considered in [8] is a natural generalization of the class of locally finite division rings. In Section 2, we shall prove that a division ring is weakly locally finite if and only if it is locally PI. On the other hand, in [7], Zhang gives one example of a locally PI division ring whose GKdim is 2. Therefore, the class of weakly locally finite division rings strictly contains the class of locally finite division rings. Also, Zhang has noted that it is unknown if there exists a locally PI division ring  $D$  with

center  $F$  whose GKdim is a non-integer [7]. Our main purpose in this paper is to construct for a given  $n$  which is either non-negative integer or  $n = \infty$  a weakly locally finite division ring whose GKdim is  $n$ . Moreover, we show that the GKdim of any weakly locally finite division ring is either non-negative integer or infinite. Therefore, in view of Theorem 2.2 in Section 2, this gives the definite answer to Zhang's question we have mentioned above. The second purpose is to study some questions related with the Kurosh Problem for division rings. Recall that in 1941, Kurosh [16, Problem R] asked if a finitely generated algebraic algebra is necessarily a finite dimensional vector space over a base field. This is a ring-theoretic analogue of the famous Burnside Problem in Group Theory: whether a finitely generated group whose elements all have finite order is necessarily finite. Both the Kurosh Problem and the Burnside Problem were solved negatively by Golod and Shafarevich [11]-[12]. In fact, they constructed an example of an infinite finitely generated group in which every element has finite order as well as one of an infinite dimensional finitely generated algebraic algebra. Concerning the Kurosh Problem, as he remarked, the particular case [16, Problem K] of division rings is of special interest. Moreover, Rowen [22] pointed out that, in general, there are two special cases we have to consider: the case of nil rings and the case of division rings. The problem has the negative answer for nil rings: there are several valuable examples [25]-[27], [18], [3] by Smoktunovicz and others of infinite dimensional finitely generated algebraic nil rings. However, for division rings, the problem remains without definite answer and this case is usually referred as the Kurosh Problem for division rings. For an additional information about this problem we refer to [14] and [28]. Clearly, a locally finite division ring is algebraic, and the converse is equivalent to the Kurosh Problem. More exactly, the Kurosh Problem for division rings will be answered in the affirmative iff any algebraic division ring is locally finite. At the present, this problem remains still unsolved in general: there are no similar examples as in the case of nil rings on one side, and on the other side, it is answered in the affirmative for all known special cases of a division ring  $D$  with the center  $F$ . In particular, it is the case: for  $F$  uncountable [22], for  $F$  finite [17], and for  $F$  having only finite algebraic field extensions (in particular for  $F$  algebraically closed). The last case follows from the Levitzki-Shirshov theorem which states that *any algebraic algebra of bounded degree is locally finite* (see e.g. [6], [14]). The answer for the case of finite  $F$  is due to Jacobson who proved that an *algebraic division ring  $D$  is commutative provided its center is finite* (see, for example, [17]). In the present paper, we investigate the Kurosh Problem and some its weaker versions for matrix rings over a division ring. In particular, we prove that the Kurosh Problem is true for weakly locally finite division rings.

Finally, we shall show that one conjecture of Herstein is also solved in the affirmative for weakly locally finite division rings.

## 2. WEAKLY LOCALLY FINITE DIVISION RINGS

Let  $D$  be a division ring with center  $F$ . Recall that  $D$  is *centrally finite* if  $D$  is a finite dimensional vector space over  $F$ . If for every finite subset  $S$  of  $D$ , the division subring  $F(S)$  generated by  $S$  over  $F$  is a finite dimensional vector space over  $F$  then  $D$  is called *locally finite*. We begin with the observation that in a centrally finite division ring, every division subring is itself centrally finite. Using this fact, it is easy to show that in a locally finite division ring, every finite subset

generates a centrally finite division subring. Motivating by this observation, we have introduced the following notion.

**Definition 2.1.** We say that a division ring  $D$  is *weakly locally finite* if for every finite subset  $S$  of  $D$ , the division subring generated by  $S$  in  $D$  is centrally finite.

Recall that an algebra  $A$  over a field  $k$  is said to be a *locally PI algebra* if every finitely generated subalgebra of  $A$  is a PI algebra. It turns out that weakly locally finite division rings are exactly locally PI division rings (regarding as algebras over their centers) as the following theorem shows.

**Theorem 2.2.** *A division ring is weakly locally finite if and only if it is locally PI.*

*Proof.* Let  $D$  be division ring with center  $F$ .

( $\Rightarrow$ ) Assume that  $D$  is weakly locally finite. For any finite subset  $S$  of  $D$ , we have to prove that the subring  $F[S]$  of  $D$  generated by  $S$  over  $F$  is a PI algebra. Indeed, the division subring  $L$  of  $D$  generated by  $S$  is centrally finite. Let  $S = \{s_1, s_2, \dots, s_t\}$ , and  $\mathcal{B} = \{x_1, x_2, \dots, x_n\}$  be a basis of  $L$  over its center  $Z(L)$ . For any  $1 \leq i, j \leq t$ , write  $s_i s_j = a_{ij1}x_1 + a_{ij2}x_2 + \dots + a_{ijn}x_n$ , where  $a_{ijk} \in Z(L)$ . Then, the division subring  $K$  of  $D$  generated by  $F$  and all  $a_{ijk}$  is a subfield of  $D$ .

Put  $H = K[S]$ . Then,  $H$  is a subring of  $D$  containing  $F[S]$ , the field  $K$  is contained in the center of  $H$  and  $H$  is a finite dimensional vector space over  $K$ . Hence,  $H$  can be considered as a subring of the matrix ring  $M_m(K)$  with  $m = \dim_K H$ . Since  $M_m(K)$  is a PI algebra,  $H$  is a PI algebra as well. Therefore,  $F[S]$  is a PI algebra.

( $\Leftarrow$ ) Assume that  $D$  is a locally PI algebra and  $S$  is a finite subset of  $D$ . Then,  $F[S]$  is a PI  $F$ -algebra. By [7, Theorem 5.6],  $F[S]$  is an Ore domain. In view of the Posner Theorem [2, Theorem 6.1.11], the division subring  $F(S)$  of  $D$  generated by  $S$  over  $F$  is centrally finite, so is every its division subring. In particular,  $\langle S \rangle$  is centrally finite. Therefore,  $D$  is weakly locally finite.  $\square$

### 3. THE GELFAND-KIRILLOV DIMENSION OF AN ALGEBRA

Let  $A$  be an algebra over a field  $k$  and  $V$  be a subspace of  $A$  containing the identity  $1_A$  of  $A$ . Assume that  $V$  is generated by elements  $a_1, a_2, \dots, a_n$ . For any integer  $r \geq 2$ ,  $V^r$  denotes a subspace of  $A$  generated by all monomials  $a_{i_1}a_{i_2}\dots a_{i_r}$  of length  $r$ , where  $a_{i_j} \in \{a_1, a_2, \dots, a_n\}$ . If  $\sum_{r \geq 1} V^r = A$ , then we say that  $V$  is a *subframe* of  $A$ . The Gelfand-Kirillov dimension of  $A$  over  $k$ , denoted by  $\text{GKdim}_k(A)$ , is defined by the following formulae

$$\text{GKdim}_k(A) := \sup_V \overline{\lim}_{r \rightarrow \infty} \log_r \dim_k V^r,$$

where  $V$  runs over the set of all subframes of  $A$ . The basic properties of the Gelfand-Kirillov dimension can be found in [15]. If  $A$  is finite dimensional over  $k$ , then  $\text{GKdim}_k(A) = 0$ . More generally, one can show that  $A$  is locally finite if and only if  $\text{GKdim}_k(A) = 0$ . Also, it is known that every positive integer can occur as the Gelfand-Kirillov dimension of some commutative PI algebra. Moreover, for every real number  $r \geq 2$ , there always exists some  $k$ -algebra  $A$  with  $\text{GKdim}_k(A) = r$ . Bergman [4] proved that there does not exist any  $k$ -algebra having Gelfand-Kirillov dimension in the open interval  $(1, 2)$ .

For a division ring  $D$ , its Gelfand-Kirillov dimension is understood the Gelfand-Kirillov dimension of an algebra  $D$  over its center  $F$ .

Recall that in [7], Zhang defined the Gelfand-Kirillov transcendence degree of an algebra  $A$  over a field  $k$  by

$$\text{Tdeg}_k A = \sup_V \inf_b \overline{\lim}_{n \rightarrow \infty} \log_n \dim_k((k + bV)^n),$$

where  $V$  ranges over subframes of  $A$  and  $b$  ranges over  $A^* = A \setminus \{0\}$ .

From [7, Theorem 5.6] and Theorem 2.2, it follows that if  $D$  is a weakly locally finite division ring with center  $F$ , then  $\text{Tdeg}_F D = \text{GKdim}_F D$ . Zhang have noted [7, Page 2871] that it is unknown if there exists a division ring  $D$  with center  $F$  such that  $\text{Tdeg}_F D$  is a non-integer. We shall prove that there are no weakly locally finite division ring whose Gelfand-Kirillov dimension is non-integer.

**Theorem 3.1.** *Let  $D$  be a weakly locally finite division ring with center  $F$ . Then,  $\text{GKdim}_F D \in \mathbb{N}$  or  $\text{GKdim}_F D = \infty$ .*

*Proof.* By Theorem 2.2,  $D$  is a locally PI algebra. By a remark in [15, Page 14],

$$\text{GKdim}_F D = \max\{\text{GKdim}_F B \mid B \subseteq D, B \text{ is finitely generated over } F\}.$$

Therefore, to prove the theorem, it suffices to show that any  $F$ -subalgebra  $B$  of  $D$  generated by a finite subset  $S$ , the Gelfand-Kirillov dimension  $\text{GKdim}_F B$  is an integer. Indeed, since  $D$  is a locally PI  $F$ -algebra,  $B$  is a PI  $F$ -algebra. Now, in view of [15, Theorem 10.5],  $\text{GKdim}_F B$  is an integer.  $\square$

#### 4. EXAMPLES

As we have noted in the Introduction, we devote an important part of this paper to construct for a given  $n$  which is either non-negative integer or  $n = \infty$  a weakly locally finite division ring whose  $\text{GKdim}$  is  $n$ . Recall that a division ring is locally finite iff its  $\text{GKdim}$  is 0 [15]. On the other hand, there exists a vast number of locally finite division rings, so the case  $n = 0$  is not necessarily considered here.

**4.1. Weakly locally finite division ring with the Gelfand-Kirillov dimension one.** The case  $n = 1$  is more complicated than the cases  $n \geq 2$  or  $n = \infty$ , so here, we consider it separately.

**Theorem 4.1.** *There exists a weakly locally finite division ring having Gelfand-Kirillov dimension one which is not algebraic over its center.*

*Proof.* Denote by  $G = \bigoplus_{i=1}^{\infty} \mathbb{Z}$  the direct sum of infinitely many copies of the additive group  $\mathbb{Z}$ . For any positive integer  $i$ , denote by  $x_i = (0, \dots, 0, 1, 0, \dots)$  the element of  $G$  with 1 in the  $i$ -th position and 0 elsewhere. Then  $G$  is a free abelian group generated by all  $x_i$  and every element  $x \in G$  is written uniquely in the form

$$x = \sum_{i \in I} n_i x_i, \tag{1}$$

with  $n_i \in \mathbb{Z}$  and some finite set  $I$ .

Now, we define an order in  $G$  as follows:

For elements  $x = (n_1, n_2, n_3, \dots)$  and  $y = (m_1, m_2, m_3, \dots)$  in  $G$ , define  $x < y$  if either  $n_1 < m_1$  or there exists  $k \in \mathbb{N}$  such that  $n_1 = m_1, \dots, n_k = m_k$  and  $n_{k+1} < m_{k+1}$ . Clearly, with this order  $G$  is a totally ordered set.

Suppose that  $p_1 < p_2 < \dots < p_n < \dots$  is a sequence of prime numbers and  $K = \mathbb{Q}(\sqrt{p_1}, \sqrt{p_2}, \dots)$  is the subfield of the field  $\mathbb{R}$  of real numbers generated by  $\mathbb{Q}$

and  $\sqrt{p_1}, \sqrt{p_2}, \dots$ , where  $\mathbb{Q}$  is the field of rational numbers. For any  $i \in \mathbb{N}$ , suppose that  $f_i : K \rightarrow K$  is a  $\mathbb{Q}$ -isomorphism satisfying the following condition:

$$f_i(\sqrt{p_i}) = -\sqrt{p_i}; \quad \text{and} \quad f_i(\sqrt{p_j}) = \sqrt{p_j} \quad \text{for any } j \neq i.$$

It is easy to verify that  $f_i f_j = f_j f_i$  for any  $i, j \in \mathbb{N}$ .

- *Step 1. Proving that, for  $x \in K$ ,  $f_i(x) = x$  for any  $i \in \mathbb{N}$  if and only if  $x \in \mathbb{Q}$ :*

The converse is obvious. Now, suppose that  $x \in K$  such that  $f_i(x) = x$  for any  $i \in \mathbb{N}$ . By setting  $K_0 = \mathbb{Q}$  and  $K_i = \mathbb{Q}(\sqrt{p_1}, \dots, \sqrt{p_i})$  for  $i \geq 1$ , we have the following ascending series:

$$K_0 \subset K_1 \subset \dots \subset K_i \subset \dots$$

If  $x \notin \mathbb{Q}$ , then there exists  $i \geq 1$  such that  $x \in K_i \setminus K_{i-1}$ . So, we have  $x = a + b\sqrt{p_i}$ , with  $a, b \in K_{i-1}$  and  $b \neq 0$ . Since  $f_i(x) = x$ ,  $0 = x - f_i(x) = 2b\sqrt{p_i}$ , a contradiction.

- *Step 2. Constructing a Laurent series ring:*

For any  $x = (n_1, n_2, \dots) = \sum_{i \in I} n_i x_i \in G$ , define  $\Phi_x := \prod_{i \in I} f_i^{n_i}$ . Clearly  $\Phi_x \in \text{Gal}(K/\mathbb{Q})$  and the map  $\Phi : G \rightarrow \text{Gal}(K/\mathbb{Q})$ , defined by  $\Phi(x) = \Phi_x$  is a group homomorphism. The following conditions hold.

- i)  $\Phi(x_i) = f_i$  for any  $i \in \mathbb{N}$ .
- ii) If  $x = (n_1, n_2, \dots) \in G$ , then  $\Phi_x(\sqrt{p_i}) = (-1)^{n_i} \sqrt{p_i}$ .

For the convenience, from now on we write the operation in  $G$  multiplicatively. For  $G$  and  $K$  as above, consider formal sums of the form

$$\alpha = \sum_{x \in G} a_x x, \quad a_x \in K.$$

For such an  $\alpha$ , define the support of  $\alpha$  by  $\text{supp}(\alpha) = \{x \in G : a_x \neq 0\}$ . Put

$$D = K((G, \Phi)) := \left\{ \alpha = \sum_{x \in G} a_x x, a_x \in K \mid \text{supp}(\alpha) \text{ is well-ordered} \right\}.$$

For  $\alpha = \sum_{x \in G} a_x x$  and  $\beta = \sum_{x \in G} b_x x$  from  $D$ , define

$$\alpha + \beta = \sum_{x \in G} (a_x + b_x) x; \quad \text{and} \quad \alpha\beta = \sum_{z \in G} \left( \sum_{xy=z} a_x \Phi_x(b_y) \right) z.$$

With operations defined as above,  $D = K((G, \Phi))$  is a division ring (we refer to [6, pp. 243-244]). Moreover, the following conditions hold.

- iii) For any  $x \in G, a \in K$ ,  $xa = \Phi_x(a)x$ .
- iv) For any  $i \neq j$ ,  $x_i \sqrt{p_i} = -\sqrt{p_i} x_i$  and  $x_j \sqrt{p_i} = \sqrt{p_i} x_j$ .
- v) For any  $i \neq j$  and  $n \in \mathbb{N}$ ,  $x_i^n \sqrt{p_i} = (-1)^n \sqrt{p_i} x_i^n$  and  $x_j^n \sqrt{p_i} = \sqrt{p_i} x_j^n$ .

- *Step 3. Finding the center of  $D$ :*

Put  $H = \{x^2 \mid x \in G\}$  and  $\mathbb{Q}((H)) = \left\{ \alpha = \sum_{x \in H} a_x x, a_x \in \mathbb{Q} \mid \text{supp}(\alpha) \text{ is well-ordered} \right\}$ .

It is easy to check that  $H$  is a subgroup of  $G$  and for every  $x \in H$ ,  $\Phi_x = \text{Id}_K$ .

Denote by  $F$  the center of  $D$ . We claim that  $F = \mathbb{Q}((H))$ . Suppose that  $\alpha = \sum_{x \in H} a_x x \in \mathbb{Q}((H))$ . Then, for every  $\beta = \sum_{y \in G} b_y y \in D$ , we have  $\Phi_x(b_y) = b_y$  and

$\Phi_y(a_x) = a_x$ . Hence

$$\begin{aligned}\alpha\beta &= \sum_{z \in G} \left( \sum_{xy=z} a_x \Phi_x(b_y) \right) z = \sum_{z \in G} \left( \sum_{xy=z} a_x b_y \right) z, \\ \beta\alpha &= \sum_{z \in G} \left( \sum_{xy=z} b_y \Phi_y(a_x) \right) z = \sum_{z \in G} \left( \sum_{xy=z} a_x b_y \right) z.\end{aligned}$$

Thus,  $\alpha\beta = \beta\alpha$  for every  $\beta \in D$ , so  $\alpha \in F$ .

Conversely, suppose that  $\alpha = \sum_{x \in G} a_x x \in F$ . Denote by  $S$  the set of all elements  $x$  appeared in the expression of  $\alpha$ . Then, it suffices to prove that  $x \in H$  and  $a_x \in \mathbb{Q}$  for any  $x \in S$ . In fact, since  $\alpha \in F$ , we have  $\sqrt{p_i}\alpha = \alpha\sqrt{p_i}$  and  $\alpha x_i = x_i\alpha$  for any  $i \geq 1$ , i.e.  $\sum_{x \in S} \sqrt{p_i} a_x x = \sum_{x \in S} \Phi_x(\sqrt{p_i}) a_x x$  and  $\sum_{x \in S} a_x (xx_i) = \sum_{x \in S} \Phi_{x_i}(a_x)(xx_i)$ . Therefore, by conditions mentioned in the beginning of *Step 2*, we have  $\sqrt{p_i} a_x = \Phi_x(\sqrt{p_i}) a_x = (-1)^{n_i} \sqrt{p_i} a_x$  and  $a_x = \Phi_{x_i}(a_x) = f_i(a_x)$  for any  $x = (n_1, n_2, \dots) \in S$ . From the first equality it follows that  $n_i$  is even for any  $i \geq 1$ . Therefore  $x \in H$ . From the second equality it follows that  $a_x = f_i(a_x)$  for any  $i \geq 1$ . So by *Step 1*, we have  $a_x \in \mathbb{Q}$ . Therefore  $\alpha \in \mathbb{Q}((H))$ . Thus,  $F = \mathbb{Q}((H))$ .

• *Step 4. Suppose that  $\gamma = x_1^{-1} + x_2^{-1} + \dots$  is an infinite formal sum. Proving that  $\gamma$  is element in  $D$  not algebraic over  $F$ :*

Since  $x_1^{-1} < x_2^{-1} < \dots$ ,  $\text{supp}(\gamma)$  is well-ordered. Hence  $\gamma \in D$ . Consider the equality

$$a_0 + a_1\gamma + a_2\gamma^2 + \dots + a_n\gamma^n = 0, \quad a_i \in F. \quad (2)$$

Note that  $X = x_1^{-1}x_2^{-1}\dots x_n^{-1}$  does not appear in the expressions of  $\gamma, \gamma^2, \dots, \gamma^{n-1}$  and the coefficient of  $X$  in the expression of  $\gamma^n$  is  $n!$ . Therefore, the coefficient of  $X$  in the expression on the left side of the equality (2) is  $a_n n!$ . It follows that  $a_n = 0$ . By induction, it is easy to see that  $a_0 = a_1 = \dots = a_n = 0$ . Hence, for any  $n \in \mathbb{N}$ , the set  $\{1, \gamma, \gamma^2, \dots, \gamma^n\}$  is independent over  $F$ . Consequently,  $\gamma$  is not algebraic over  $F$ .

• *Step 5. Constructing a division subring of  $D$  which is a weakly locally finite:*

Consider the element  $\gamma$  from *Step 4*. For any  $n \geq 1$ , put

$$R_n = F(\sqrt{p_1}, \sqrt{p_2}, \dots, \sqrt{p_n}, x_1, x_2, \dots, x_n, \gamma),$$

and  $R_\infty = \bigcup_{n=1}^{\infty} R_n$ . First, we prove that  $R_n$  is centrally finite for each positive integer  $n$ . Consider the element

$$\gamma_n = x_{n+1}^{-1} + x_{n+2}^{-1} + \dots \quad (\text{infinite formal sum}).$$

Since  $\gamma_n = \gamma - (x_1^{-1} + x_2^{-1} + \dots + x_n^{-1})$ , we conclude that  $\gamma_n \in R_n$  and

$$F(\sqrt{p_1}, \sqrt{p_2}, \dots, \sqrt{p_n}, x_1, x_2, \dots, x_n, \gamma) = F(\sqrt{p_1}, \sqrt{p_2}, \dots, \sqrt{p_n}, x_1, x_2, \dots, x_n, \gamma_n).$$

Note that  $\gamma_n$  commutes with all  $\sqrt{p_i}$  and all  $x_i$  (for  $i = 1, 2, \dots, n$ ). Therefore

$$\begin{aligned}R_n &= F(\sqrt{p_1}, \sqrt{p_2}, \dots, \sqrt{p_n}, x_1, x_2, \dots, x_n, \gamma_n) \\ &= F(\gamma_n)(\sqrt{p_1}, \sqrt{p_2}, \dots, \sqrt{p_n}, x_1, x_2, \dots, x_n).\end{aligned}$$

In combination with the equalities  $(\sqrt{p_i})^2 = p_i, x_i^2 \in F, \sqrt{p_i}x_j = x_j\sqrt{p_i}, i \neq j, \sqrt{p_i}x_i = -x_i\sqrt{p_i}$ , it follows that every element  $\beta$  from  $R_n$  can be written in the

form

$$\beta = \sum_{0 \leq \varepsilon_i, \mu_i \leq 1} a_{(\varepsilon_1, \dots, \varepsilon_n, \mu_1, \dots, \mu_n)} (\sqrt{p_1})^{\varepsilon_1} \dots (\sqrt{p_n})^{\varepsilon_n} x_1^{\mu_1} \dots x_n^{\mu_n},$$

where  $a_{(\varepsilon_1, \dots, \varepsilon_n, \mu_1, \dots, \mu_n)} \in F(\gamma_n)$ .

Hence  $\beta$  is in the center of  $R_n$  if and only if  $\beta \in F(\gamma_n)$ . It means that  $F(\gamma_n)$  is the center of  $R_n$ . Moreover,  $R_n$  is a vector space over  $F(\gamma_n)$  having the finite set  $B_n$  which consists of the products

$$(\sqrt{p_1})^{\varepsilon_1} \dots (\sqrt{p_n})^{\varepsilon_n} x_1^{\mu_1} \dots x_n^{\mu_n}, 0 \leq \varepsilon_i, \mu_i \leq 1$$

as a base. Thus,  $R_n$  is centrally finite.

For any finite subset  $S \subseteq R_\infty$ , there exists  $n$  such that  $S \subseteq R_n$ . Therefore, the division subring of  $R_\infty$  generated by  $S$  over  $F$  is contained in  $R_n$ , which is centrally finite. Thus,  $R_\infty$  is weakly locally finite.

- *Step 6. Finding the center  $Z(R_\infty)$  of  $R_\infty$ :*

Since  $F \subseteq Z(R_\infty)$ , it suffices to show  $Z(R_\infty) \subseteq F$ . Consider an arbitrary  $a \in Z(R_\infty)$ . Then,  $a$  commutes with all  $x_i$  and  $\sqrt{p_i}$  for all  $i \geq 1$ . Hence,  $a$  commutes with all elements of  $G$  and  $K$ . Therefore,  $a \in Z(D) = F$ .

- *Step 7. Proving that  $R_\infty$  is not algebraic over  $F$ :*

It was shown in *Step 4* that  $\gamma \in D$  is not algebraic over  $F$ .

- *Step 8. Proving that  $\text{GKdim}_F D = 1$ :*

By [7, Lemma 5.4], it suffices to prove that  $\text{GKdim}_F R'_n = 1$ , where  $R'_n = F[\gamma, \sqrt{p_1}, \dots, \sqrt{p_n}, x_1, \dots, x_n]$  is the subring of  $R_n$  generated by

$$\gamma, \sqrt{p_1}, \dots, \sqrt{p_n}, x_1, \dots, x_n$$

over  $F$ . Let

$$V = \langle 1, \gamma, \sqrt{p_1}, \dots, \sqrt{p_n}, x_1, \dots, x_n \rangle_F$$

be the vector subspace of  $R'_n$  over  $F$ . Then  $\sum_{r \geq 1} V^r = R'_n$ , which implies that  $V$  is a subframe of  $R'_n$ . We claim that  $\dim_F V^r = M + 4^n r$  for some  $M$  and  $r$  sufficiently large. Indeed, for any  $r > 2n$ , in view of the relations between  $\sqrt{p_i}$  and  $x_j$  for any  $1 \leq i, j \leq n$  (see iv) and v) in *Step 2*), if we add the following elements

$$\gamma^{r+1}, \gamma^r \sqrt{p_1}, \dots, \gamma^r \sqrt{p_n}, \gamma^r x_1, \dots, \gamma^r x_n, \gamma^{r-1} \sqrt{p_1 p_2}, \dots, \gamma^{r-2n} \sqrt{p_1} \dots \sqrt{p_n} x_1 \dots x_n$$

to some basis of  $V^r$ , then we obtain a basis of  $V^{r+1}$ . Since the number of added elements is  $C_{2n}^0 + C_{2n}^1 + \dots + C_{2n}^{2n} = 2^{2n} = 4^n$ , one has  $\dim_F V^r = M + 4^n r$  for some integer  $M$ .

Hence,  $\overline{\lim}_{r \rightarrow \infty} \log_r \dim_F V^r = \overline{\lim}_{r \rightarrow \infty} \log_r (M + 4^n r) = 1$ . Now using the remark in [15, page 14], we have  $\text{GKdim}_F R'_n = \overline{\lim}_{r \rightarrow \infty} \log_r \dim_F V^r = 1$ .

The proof of the theorem is now complete.  $\square$

**Remark 1.** Note that in [23], it is shown that every algebra over a field with the Gelfand-Kirillov dimension equal to 1 is locally PI. Hence, in view of Theorem 2.2, the fact proved in *Step 8* confirms again that the division ring we have constructed in *Step 5* is weakly locally finite.

**4.2. Weakly locally finite division ring with the Gelfand-Kirillov dimension  $n \geq 2$ .** Throughout this subsection,  $k = \mathbb{C}$  is the field of real numbers,  $n$  is a given positive integer and  $p$  is a prime number. For positive integers  $t$ , we construct a sequence of  $k$ -algebras  $A_{nt}$  as the following:

Let  $\{x_{11}, x_{2t}, \dots, x_{nt}\}$  be  $n$  non-commutative indeterminates and  $r_t \in \mathbb{C}$  the primitive  $p^{2t}$ -th root of unity such that  $r_t = r_{t+1}^{p^2}$ . Consider

$$A_{nt} = k\langle x_{11}, x_{2t}, \dots, x_{nt} \rangle / \langle x_{it}x_{jt} - r_t x_{jt}x_{it} \mid i \neq j \rangle,$$

where  $k\langle x_{11}, x_{2t}, \dots, x_{nt} \rangle$  is the free algebra in  $\{x_{11}, x_{2t}, \dots, x_{nt}\}$  over  $k$  and  $\langle x_{it}x_{jt} - r_t x_{jt}x_{it} \mid i \neq j \rangle$  is the ideal in  $k\langle x_{11}, x_{2t}, \dots, x_{nt} \rangle$  generated by all  $x_{it}x_{jt} - r_t x_{jt}x_{it}$  with  $i \neq j$ . For an element  $\alpha \in k\langle x_{11}, x_{2t}, \dots, x_{nt} \rangle$ , the symbol  $\bar{\alpha}$  denotes the image of  $\alpha$  via the natural  $k$ -homomorphism  $k\langle x_{11}, x_{2t}, \dots, x_{nt} \rangle \rightarrow A_{nt}$ . Since for any  $i \neq j$ ,  $x_{it}x_{jt} - r_t x_{jt}x_{it}$  is irreducible,  $A_{nt}$  is a domain.

**Lemma 4.2.** *The following statements hold:*

- (1)  $\text{GKdim}_k A_{nt} = n$ .
- (2)  $A_{nt}$  is an Ore domain, the quotient division ring  $D(A_{nt})$  of  $A_{nt}$  is weakly locally finite, and  $\text{GKdim}_k D(A_{nt}) = n$ .

*Proof.* (1) Since  $A_{nt}$  is finitely generated over  $k$ , by a remark in [7, page 14],  $\text{GKdim}_k A_{nt} = \overline{\lim}_{r \rightarrow \infty} \log_r \dim_k V^r$  for any subframe  $V$  of  $A_{nt}$ . Let

$$V = \langle \bar{1}, \bar{x}_{1t}, \bar{x}_{2t}, \dots, \bar{x}_{nt} \rangle_k$$

be the vector subspace of  $A_{nt}$  generated by  $\{\bar{1}, \bar{x}_{1t}, \bar{x}_{2t}, \dots, \bar{x}_{nt}\}$  over  $k$ . It is clear that  $V$  is a subframe of  $A_{nt}$ . Now, we have  $V^r = \langle \bar{f} \mid f \in B_r \rangle_k$ , where  $B_r$  is the set of monomials  $\bar{x}_{j_1 t} \bar{x}_{j_2 t} \dots \bar{x}_{j_s t}$  of length  $s \leq r$  and  $j_i \leq j_{i+1}$  (notice that the monomial of length 0 is  $\bar{1}$ ). Observe that  $B_r$  is independent over  $k$  and the cardinality of  $B_r$  is

$$C_{n-1}^0 + C_{n+1-1}^1 + C_{n+2-1}^2 + \dots + C_{n+r-1}^r = C_{n+r}^{r+1}.$$

This means that  $\dim_k V^r = C_{n+r}^{r+1}$ . Hence,

$$\text{GKdim}_k A_{nt} = \overline{\lim}_{r \rightarrow \infty} \log_r \dim_k V^r = \overline{\lim}_{r \rightarrow \infty} \log_r C_{n+r}^{r+1} = n.$$

(2) Since  $\text{GKdim}_k A_{nt} = n$ , the algebra  $A_{nt}$  is an Ore domain by [7, propositions 3.2 and 2.1]. Denote by  $D_{nt} = D(A_{nt})$  the quotient division ring of  $A_{nt}$ . We will prove that  $D_{nt}$  is weakly locally finite. Indeed, put  $k_t = k(\bar{x}_{1t}^{p^{2t}}, \bar{x}_{2t}^{p^{2t}}, \dots, \bar{x}_{nt}^{p^{2t}})$ , the division subring of  $D_{nt}$  generated by  $\bar{x}_{1t}^{p^{2t}}, \bar{x}_{2t}^{p^{2t}}, \dots, \bar{x}_{nt}^{p^{2t}}$  over  $k$ , one has  $x_{it}x_{jt}^{p^{2t}} = r_t^{p^{2t}} x_{jt}^{p^{2t}} x_{it} = x_{jt}^{p^{2t}} x_{it}$  for any  $1 \leq i, j \leq n$ , which implies that  $\bar{x}_{it}^{p^{2t}}$  belongs to the center  $Z(D_{nt})$  of  $D_{nt}$ , and, consequently,  $k_t$  is contained in  $Z(D_{nt})$ . Let  $B_{nt} = k_t[\bar{x}_{1t}, \bar{x}_{2t}, \dots, \bar{x}_{nt}]$  be the subring of  $D_{nt}$  generated by  $\bar{x}_{11}, \bar{x}_{2t}, \dots, \bar{x}_{nt}$  over  $k_t$ . Then  $A_{nt} \subseteq B_{nt}$ . Observe that every element  $f$  of  $B_{nt}$  is of the form

$$f = \sum_{0 \leq \mu_i \leq p^{2t}} a_{(\mu_1, \dots, \mu_n)} \bar{x}_{1t}^{\mu_1} \dots \bar{x}_{nt}^{\mu_n},$$

where  $a_{(\mu_1, \dots, \mu_n)} \in k_t$ . This implies that  $B_{nt}$  is a finite-dimensional vector space over  $k_t$ . Thus,  $B_{nt}$  can be considered as a subring of  $M_m(k_t)$  with  $m = \dim_{k_t} B_{nt}$ . Since  $M_m(k_t)$  is a PI-algebra, so are  $B_{nt}$  and  $A_{nt}$ . Applying [7, Theorem 5.6], one has  $D_{nt}$  is locally PI and  $\text{GKdim}_k D_{nt} = n$ . Now using Theorem 2.2, we have  $D_{nt}$  is weakly locally finite.  $\square$



For any pair  $(n, t)$  of positive integers, consider the  $k$ -homomorphism

$$\phi_{nt}: k\langle x_{1(t-1)}, x_{2(t-1)}, \dots, x_{n(t-1)} \rangle \rightarrow k\langle x_{1t}, x_{2t}, \dots, x_{nt} \rangle,$$

defined by  $\phi_{nt}(x_{i(t-1)}) = x_{it}^p$  for any  $1 \leq i \leq n$ . Then,  $\phi_{nt}$  induces a  $k$ -homomorphism from  $A_{n,t-1}$  to  $A_{nt}$ . In fact, we have the following lemma.

**Lemma 4.3.** *The  $k$ -homomorphism  $\phi_{nt}$  induces the injective  $k$ -homomorphism*

$$\Phi_{nt}: A_{n,t-1} \rightarrow A_{nt},$$

with  $\Phi_{nt}(\overline{x_{i(t-1)}}) = \overline{x_{it}^p}$ , for any  $1 \leq i \leq n$ .

*Proof.* The most important thing in the proof of this lemma is to check that  $\Phi$  is well-defined. To do this, we will show that for any  $1 \leq i \neq j \leq n$ , the image  $\phi(x_{i(t-1)}x_{j(t-1)} - r_{t-1}x_{j(t-1)}x_{i(t-1)}) \in \langle x_{it}x_{jt} - r_t x_{jt}x_{it} \mid 1 \leq i \neq j \leq n \rangle$ . Indeed, for any  $1 \leq i \neq j \leq n$ , we have

$$\begin{aligned} \phi(x_{i(t-1)}x_{j(t-1)} - r_{t-1}x_{j(t-1)}x_{i(t-1)}) &= x_{it}^p x_{jt}^p - r_{t-1} x_{jt}^p x_{it}^p = x_{it}^p x_{jt}^p - r_t^{p^2} x_{jt}^p x_{it}^p \\ &= x_{it}^{p-1} (x_{it} x_{jt} - r_t x_{jt} x_{it}) x_{jt}^{p-1} \in \langle x_{it} x_{jt} - r_t x_{jt} x_{it} \mid 1 \leq i \neq j \leq n \rangle. \end{aligned}$$

□

Now, for a given positive integer  $n$ , we are ready to give an example of a division ring with the Gelfand-Kirillov dimension  $n$ .

**Theorem 4.4.** *Let  $A_n = \bigcup_{t \geq 1} A_{nt}$ . Then,  $A_n$  is an Ore domain. Moreover, if*

*$D_n = D(A_n)$  is the quotient division ring of  $A_n$ , then we have*

- (1) *The center  $Z(D_n)$  of  $D_n$  is  $k$ .*
- (2)  *$D_n = \bigcup_{t \geq 1} D_{nt}$  is weakly locally finite.*
- (3)  *$\text{GKdim}_k D_n = n$ .*
- (4)  *$D_n$  is not algebraic over  $Z(D_n)$ . In particular,  $D_n$  is not locally finite.*

*Proof.* The algebra  $A_n$  is an Ore domain by [7, Lemma 5.4 (2)].

(1) Since  $k \subseteq Z(D_n)$ , we have to show  $Z(D_n) \subseteq k$ . It suffices to show that none of indeterminates  $x_{is}$  occurs in  $f$  for any  $f \in Z(D_n)$ . Assume that this is false. Without loss of generality, we can suppose that  $x_{1s}^m$  occurs in  $f$ , where  $m$  is a non-zero integer with the smallest absolute value. Since  $f \in Z(D_n) \subseteq D = \bigcup_{t \geq 1} D_{nt}$ ,

there exists a positive integer  $t_0$  such that  $f \in D_{nt_0}$ . Hence,  $f \in D_{nt}$  for any  $t \geq t_0$ , so that  $f \in Z(D_{nt})$  for any  $t \geq t_0$ . Using arguments as in the proof of Lemma 4.2, we conclude that  $k_t = k(\overline{x_{1t}^{p^{2t}}}, \overline{x_{2t}^{p^{2t}}}, \dots, \overline{x_{nt}^{p^{2t}}})$  is the center of  $D_{nt}$ . In view of Lemma 4.3, the element  $\overline{x_{1s}^m} \in A_{ns}$  can be considered as the element  $\overline{x_{1t}^{mp^{t-s}}}$  in  $A_{nt}$  (via homomorphisms  $\Phi_{nt}$ ) for any  $t \geq s$ . Since  $f \in k_t$  for any  $t \geq \max\{t_0, s\}$ , all powers of  $\overline{x_{1t}}$  divide  $p^{2t}$  for any such a  $t$ . In particular,  $mp^{t-s}$  divides  $p^{2t}$  for any  $t \geq \max\{t_0, s\}$ , which is a contradiction.

(2) We have  $D_n = \bigcup_{t \geq 1} D_{nt}$  by [7, Lemma 5.4 (2)]. Now for any finite subset  $G$  of  $D_n$ , there exists  $t_G$  such that  $G \subseteq D_{nt_G}$ . Thus, the division subring of  $D_n$  generated by  $G$  is contained in  $D_{nt_G}$ , hence it is centrally finite by Lemma 4.2. Therefore,  $D_n$  is weakly locally finite.

(3) In view of (2) together with [7, Theorem 5.6 and Lemma 5.4 (1)], it follows that  $\text{GKdim}_k D_n = n$ .

(4) The conclusion is clear since  $x_{11}$  is not algebraic over  $Z(D_n) = k$ . □

**4.3. Weakly locally finite division ring with the infinite Gelfand-Kirillov dimension.** For any pair  $(n, t)$  of positive integers, consider the  $k$ -homomorphism

$$\psi_{nt}: k\langle x_{1t}, x_{2t}, \dots, x_{nt} \rangle \rightarrow k\langle x_{1t}, x_{2t}, \dots, x_{n+1,t} \rangle,$$

defined by  $\psi_{nt}(x_{nj}) = x_{n+1,j}^p$  for any  $1 \leq j \leq t$ . Then,  $\psi_{nt}$  induces a  $k$ -homomorphism from  $A_{nt}$  to  $A_{n+1,t}$ . In fact, we have the following lemma.

**Lemma 4.5.** *The  $k$ -homomorphism  $\psi_{nt}$  induces the injective  $k$ -homomorphism*

$$\Psi_{nt}: A_{nt} \rightarrow A_{n+1,t},$$

with  $\Psi_{nt}(\overline{x_{nj}}) = \overline{x_{n+1,j}^p}$ , for any  $1 \leq j \leq t$ .

*Proof.* The proof of this lemma is similar to the proof of Lemma 4.3.  $\square$

Put  $A = \bigcup_{n \geq 1} A_n$ . Then, in view of [7, Lemma 5.4],  $A$  is an Ore domain, and  $D = \bigcup_{n \geq 1} D_n$  is the quotient division ring of  $A$ . By the same arguments as in the proof of Theorem 4.4, we get the following result.

**Theorem 4.6.** *Let  $A$  and  $D$  be as above. Then, the following statements hold:*

- (1) *The center  $Z(D)$  of  $D$  is  $k$ .*
- (2)  *$D$  is weakly locally finite.*
- (3)  *$\text{GKdim}_k D = \infty$ .*
- (4)  *$D$  is not algebraic over  $Z(D)$ . In particular,  $D$  is not locally finite.*

*Proof.* The proofs of (1), (2) and (4) are similar to that in the proof of Theorem 4.4. So, it remains to prove (3). In fact, we have  $\text{GKdim}_k D \geq \text{GKdim}_k D_n = n$  for any  $n$  by Theorem 4.4. Hence,  $\text{GKdim}_k D = \infty$ .  $\square$

## 5. SOME FACTS RELATED WITH THE KUROSH PROBLEM

The Kurosh Problem for division rings we have mentioned in the Introduction can be formulated as the following.

**Problem 5.1.** *Is it true that every algebraic division ring is locally finite?*

As we have observed in Section 2, in a locally finite division ring, every finite subset generates a centrally finite division subring. Therefore, by definition, every locally finite division ring is weakly locally finite. Zhang has constructed one example [7, Example 5.7] of a locally PI division ring which is not algebraic over the center. Therefore, in view of Theorem 2.2, we see that the class of weakly locally finite division rings strictly contains the class of locally finite division rings.

The following theorem shows that the Kurosh problem is solved in the affirmative for the class of weakly locally finite division rings.

**Theorem 5.2.** *A division ring  $D$  is locally finite if and only if  $D$  is weakly locally finite and algebraic.*

*Proof.* If  $D$  is locally finite, then clearly  $D$  is both weakly locally finite and algebraic. Conversely, assume that  $D$  is both weakly locally finite and algebraic. Let  $F = Z(D)$  and  $S$  be a finite subset of  $D$ . Since  $D$  is weakly locally finite, the division subring  $L$  of  $D$  generated by  $S$  is centrally finite. Let  $\mathcal{B} = \{x_1, x_2, \dots, x_n\}$  be the basis of  $[L : Z(L)]$ . For any  $1 \leq i, j \leq n$ , write  $x_i x_j = a_{ij1}x_1 + a_{ij2}x_2 + \dots + a_{ijn}x_n$ ,

where  $a_{ijk} \in Z(L)$ . Let  $K$  be the division subring of  $D$  generated by  $F$  and all  $a_{ijk}$ . One has  $K$  is a subfield of  $D$ . By  $D$  is algebraic over  $F$  and set of all  $a_{ijk}$  is finite,  $K/F$  is a finite field extension.

Let  $H = \{a_1x_1 + \dots + a_nx_n \mid a_i \in K\}$ . Then  $H$  is a finite dimensional vector space over  $K$ , and it is clear that  $H$  is a subring of  $D$ . Now, for any  $x \in H$ , the set  $\{1, x, x^2, \dots, x^{n+1}\}$  is linearly dependent over  $K$ , hence  $\sum_{i=0}^n c_i x^i = 0$  for some  $c_i \in K$  not all zero. It follows that  $x^{-1} \in H$ , so  $H$  is a division subring of  $D$ . Moreover  $\dim_F H < \infty$ , since  $\dim_K H < \infty$  and  $K/F$  is a finite field extension.

It is easy to see that  $H = F(S)$ , and the proof is now complete.  $\square$

Note that the following weaker version of the Kurosh Problem is still open (see [28, Problem 8]).

**Problem 5.3.** *Do there exist centrally infinite finitely generated division rings?*

If  $D^*$  is finitely generated, then  $D$  is finitely generated as a division ring. The converse may not true. This fact leads us to consider the following problem which is also a weaker version of Problem 5.3.

**Problem 5.4.** *Do there exists a centrally infinite division ring  $D$  whose the multiplicative group  $D^*$  is finitely generated?*

We devote the remaining part of the present section to the study of the matrix version of the last problem. More exactly, the following problem is under our consideration.

**Problem 5.5.** *Do there exists a centrally infinite division ring  $D$  such that the group  $\mathrm{GL}_n(D)$ ,  $n \geq 1$  is finitely generated?*

In the following, we shall identify  $F^*$  with  $F^*I := \{\alpha I \mid \alpha \in F^*\}$ , where  $I$  denotes the identity matrix in  $\mathrm{GL}_n(D)$ .

In [9], we proved that if  $D$  is a division ring of type 2 and  $D^*$  is finitely generated, then  $D$  is a finite field. More generally, we showed that in a division ring of type 2, there no finitely generated non-central subgroup containing the center  $F^*$  (see [9, Theorem 2.5]). Recall that a division ring  $D$  with center  $F$  is of *type 2* if for every two elements  $x, y \in D$ , the division subring  $F(x, y)$  is a finite dimensional vector space over  $F$ .

In [19, Theorem 1], it was proved that if  $D$  is centrally finite, then any finitely generated subnormal subgroup of  $D^*$  is central. This result can be carried over for weakly locally finite division rings as the following.

**Theorem 5.6.** *Let  $D$  be a weakly locally finite division ring. Then, every finitely generated subnormal subgroup of  $D^*$  is central.*

*Proof.* Since  $N$  is finitely generated and  $D$  is weakly locally finite, the division subring generated by  $N$ , namely  $L$ , is centrally finite. By [19, Theorem 1],  $N \subseteq Z(L)$ . Consequently,  $N$  is abelian. Now, by [24, 14.4.4, p. 440],  $N \subseteq Z(D)$ .  $\square$

The following theorem is a generalization of [1, Theorem 5].

**Theorem 5.7.** *Let  $D$  be a weakly locally finite division ring with center  $F$  and  $N$  be an infinite subnormal subgroup of  $\mathrm{GL}_n(D)$ ,  $n \geq 2$ . If  $N$  is finitely generated, then  $N \subseteq F$ .*

*Proof.* Suppose that  $N$  is non-central. Then, by [20, Theorem 11],  $\mathrm{SL}_n(D) \subseteq N$ . So,  $N$  is normal in  $\mathrm{GL}_n(D)$ . Suppose that  $N$  is generated by matrices  $A_1, A_2, \dots, A_k$  in  $\mathrm{GL}_n(D)$  and  $T$  is the set of all coefficients of all  $A_j$ . Since  $D$  is weakly locally finite, the division subring  $L$  generated by  $T$  is centrally finite. It follows that  $N$  is a normal finitely generated subgroup of  $\mathrm{GL}_n(L)$ . By [1, Theorem 5],  $N \subseteq Z(\mathrm{GL}_n(L))$ . In particular,  $N$  is abelian and consequently,  $\mathrm{SL}_n(D)$  is abelian, a contradiction.  $\square$

**Lemma 5.8.** *Let  $D$  be a division ring and  $n \geq 1$ . Then,  $Z(\mathrm{SL}_n(D))$  is a torsion group if and only if  $Z(D')$  is a torsion group.*

*Proof.* The case  $n = 1$  is clear. So, we can assume that  $n \geq 2$ . Denote by  $F$  the center of  $D$ . By [5, §21, Theorem 1, p.140],

$$Z(\mathrm{SL}_n(D)) = \{dI \mid d \in F^* \text{ and } d^n \in D'\}.$$

If  $Z(\mathrm{SL}_n(D))$  is a torsion group, then, for any  $d \in Z(D') = D' \cap F$ ,  $dI \in Z(\mathrm{SL}_n(D))$ . It follows that  $d$  is torsion. Conversely, if  $Z(D')$  is a torsion group, then, for any  $A \in Z(\mathrm{SL}_n(D))$ ,  $A = dI$  for some  $d \in F^*$  such that  $d^n \in D'$ . It follows that  $d^n$  is torsion. Therefore,  $A$  is torsion.  $\square$

**Theorem 5.9.** *Let  $D$  be a non-commutative algebraic, weakly locally finite division ring with center  $F$  and  $N$  be a subgroup of  $\mathrm{GL}_n(D)$  containing  $F^*$ ,  $n \geq 1$ . Then  $N$  is not finitely generated.*

*Proof.* Recall that if a division ring  $D$  is weakly locally finite, then  $Z(D')$  is a torsion group (see Lemma 6.2). Therefore, by lemma 5.8,  $Z(\mathrm{SL}_n(D))$  is a torsion group.

Suppose that there is a finitely generated subgroup  $N$  of  $\mathrm{GL}_n(D)$  containing  $F^*$ . Clearly  $N/N'$  is a finitely generated abelian group, where  $N'$  denotes the derived subgroup of  $N$ . Then, in virtue of [24, 5.5.8, p. 113],  $F^*N'/N'$  is a finitely generated abelian group.

*Case 1:  $\mathrm{char}(D) = 0$ .*

Then,  $F$  contains the field  $\mathbb{Q}$  of rational numbers and it follows that  $\mathbb{Q}^*I/(\mathbb{Q}^*I \cap N') \cong \mathbb{Q}^*N'/N'$ . Since  $F^*N'/N'$  is finitely generated abelian subgroup,  $\mathbb{Q}^*N'/N'$  is finitely generated too, and consequently  $\mathbb{Q}^*I/(\mathbb{Q}^*I \cap N')$  is finitely generated. Consider an arbitrary  $A \in \mathbb{Q}^*I \cap N'$ . Then  $A \in F^*I \cap \mathrm{SL}_n(D) \subseteq Z(\mathrm{SL}_n(D))$ . Therefore  $A$  is torsion. Since  $A \in \mathbb{Q}^*I$ , we have  $A = dI$  for some  $d \in \mathbb{Q}^*$ . It follows that  $d = \pm 1$ . Thus,  $\mathbb{Q}^*I \cap N'$  is finite. Since  $\mathbb{Q}^*I/(\mathbb{Q}^*I \cap N')$  is finitely generated,  $\mathbb{Q}^*I$  is finitely generated. Therefore  $\mathbb{Q}^*$  is finitely generated, that is impossible.

*Case 2:  $\mathrm{char}(D) = p > 0$ .*

Denote by  $\mathbb{F}_p$  the prime subfield of  $F$ , we shall prove that  $F$  is algebraic over  $\mathbb{F}_p$ . In fact, suppose that  $u \in F$  and  $u$  is transcendental over  $\mathbb{F}_p$ . Put  $K := \mathbb{F}_p(u)$ , then the group  $K^*I/(K^*I \cap N')$  considered as a subgroup of  $F^*N'/N'$  is finitely generated. Considering an arbitrary  $A \in K^*I \cap N'$ , we have  $A = (f(u)/g(u))I$  for some  $f(X), g(X) \in \mathbb{F}_p[X]$ ,  $((f(X), g(X)) = 1$  and  $g(u) \neq 0$ . As mentioned above, we have  $f(u)^s/g(u)^s = 1$  for some positive integer  $s$ . Since  $u$  is transcendental over  $\mathbb{F}_p$ ,  $f(u)/g(u) \in \mathbb{F}_p$ . Therefore,  $K^*I \cap N'$  is finite and consequently,  $K^*I$  is finitely generated. It follows that  $K^*$  is finitely generated, hence  $K$  is finite. Hence  $F$  is algebraic over  $\mathbb{F}_p$  and it follows that  $D$  is algebraic over  $\mathbb{F}_p$ . Now, in virtue of Jacobson's Theorem [17, (13.11), p. 219],  $D$  is commutative, a contradiction.  $\square$

**Corollary 5.10.** *Let  $D$  be an algebraic, weakly locally finite division ring. If the group  $\mathrm{GL}_n(D)$ ,  $n \geq 1$ , is finitely generated, then  $D$  is commutative.*

If  $M$  is a maximal finitely generated subgroup of  $\mathrm{GL}_n(D)$ , then  $\mathrm{GL}_n(D)$  is finitely generated. So, the next result follows immediately from Corollary 5.10.

**Corollary 5.11.** *Let  $D$  be an algebraic, weakly locally finite division ring. If the group  $\mathrm{GL}_n(D)$ ,  $n \geq 1$ , has a maximal finitely generated subgroup, then  $D$  is commutative.*

By the same way as in the proof of Theorem 5.9, we obtain the following corollary.

**Corollary 5.12.** *Let  $D$  be a non-commutative algebraic, weakly locally finite division ring with center  $F$  and  $S$  is a subgroup of  $\mathrm{GL}_n(D)$ . If  $N = F^*S$ , then  $N/N'$  is not finitely generated.*

*Proof.* Suppose that  $N/N'$  is finitely generated. Since  $N' = S'$  and  $F^*I/(F^*I \cap S') \cong F^*S'/S'$ ,  $F^*I/(F^*I \cap S')$  is a finitely generated abelian group. Now, by the same arguments as in the proof of Theorem 5.9, we conclude that  $D$  is commutative.  $\square$

**Corollary 5.13.** *Let  $D$  be a non-commutative algebraic, weakly locally finite division ring. Then,  $D^*$  is not finitely generated.*

*Proof.* Take  $N = S = \mathrm{GL}_n(D)$  in Corollary 5.12 and have in mind that

$$[\mathrm{GL}_n(D), \mathrm{GL}_n(D)] = \mathrm{SL}_n(D),$$

we see that  $D^* \cong \mathrm{GL}_n(D)/\mathrm{SL}_n(D)$  is not finitely generated.  $\square$

## 6. HERSTEIN'S CONJECTURE FOR WEAKLY LOCALLY FINITE DIVISION RINGS

Let  $K \subsetneq D$  be division rings. Recall that an element  $x \in D$  is *radical* over  $K$  if there exists some positive integer  $n(x)$  depending on  $x$  such that  $x^{n(x)} \in K$ . A subset  $S$  of  $D$  is *radical* over  $K$  if every element from  $S$  is radical over  $K$ . In 1978, I.N. Herstein [13, Conjecture 3] conjectured that given a subnormal subgroup  $N$  of  $D^*$ , if  $N$  is radical over center  $F$  of  $D$ , then  $N$  is central, i. e.  $N$  is contained in  $F$ . Herstein, himself in the cited above paper proved this fact for the special case, when  $N$  is torsion group. However, the problem remains still open in general. In [10], it was proved that this conjecture is true in the finite dimensional case. Here, we shall prove that this conjecture is also true for weakly locally finite division rings. First, we note the following two lemmas we need for our further purpose.

**Lemma 6.1.** *Let  $D$  be a division ring with center  $F$ . If  $N$  is a subnormal subgroup of  $D^*$ , then  $Z(N) = N \cap F$ .*

*Proof.* If  $N$  is contained in  $F$ , then there is nothing to prove. Thus, suppose that  $N$  is non-central. By [9, 14.4.2, p. 439],  $C_D(N) = F$ . Hence  $Z(N) \subseteq N \cap F$ . Since the inclusion  $N \cap F \subseteq Z(N)$  is obvious,  $Z(N) = N \cap F$ .  $\square$

**Lemma 6.2.** *If  $D$  is a weakly locally finite division ring, then  $Z(D')$  is a torsion group.*

*Proof.* By Lemma 6.1,  $Z(D') = D' \cap F$ . For any  $x \in Z(D')$ , there exists some positive integer  $n$  and some  $a_i, b_i \in D^*$ ,  $1 \leq i \leq n$ , such that

$$x = a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \dots a_n b_n a_n^{-1} b_n^{-1}.$$

Set  $S := \{a_i, b_i \mid 1 \leq i \leq n\}$ . Since  $D$  is weakly locally finite, the division subring  $L$  of  $D$  generated by  $S$  is centrally finite. Put  $n = [L : Z(L)]$ . Since  $x \in F$ ,  $x$  commutes with every element of  $S$ . Therefore,  $x$  commutes with every element of  $L$ , and consequently,  $x \in Z(L)$ . So,

$$x^n = N_{L/Z(L)}(x) = N_{L/Z(L)}(a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \dots a_n b_n a_n^{-1} b_n^{-1}) = 1.$$

Thus,  $x$  is torsion.  $\square$

In [13, Theorem 1], Herstein proved that if in a division ring  $D$  every multiplicative commutator  $aba^{-1}b^{-1}$  is torsion, then  $D$  is commutative. Further, with the assumption that  $D$  is a finite dimensional vector space over its center  $F$ , he proved [13, Theorem 2] that, if every multiplicative commutator in  $D$  is radical over  $F$ , then  $D$  is commutative. Now, using Lemma 6.2, we can carry over the last fact for weakly locally finite division rings.

**Theorem 6.3.** *Let  $D$  be a weakly locally finite division ring with center  $F$ . If every multiplicative commutator in  $D$  is radical over  $F$ , then  $D$  is commutative.*

*Proof.* For any  $a, b \in D^*$ , there exists a positive integer  $n = n_{ab}$  depending on  $a$  and  $b$  such that  $(aba^{-1}b^{-1})^n \in F$ . Hence, by Lemma 6.2, it follows that  $aba^{-1}b^{-1}$  is torsion. Now, by [13, Theorem 1],  $D$  is commutative.  $\square$

The following theorem gives the affirmative answer to Conjecture 3 in [13] for weakly locally finite division rings.

**Theorem 6.4.** *Let  $D$  be a weakly locally finite division ring with center  $F$  and  $N$  be a subnormal subgroup of  $D^*$ . If  $N$  is radical over  $F$ , then  $N$  is central, i.e.  $N$  is contained in  $F$ .*

*Proof.* Consider the subgroup  $N' = [N, N] \subseteq D'$  and suppose that  $x \in N'$ . Since  $N$  is radical over  $F$ , there exists some positive integer  $n$  such that  $x^n \in F$ . Hence  $x^n \in F \cap D' = Z(D')$ . By Lemma 6.2,  $x^n$  is torsion, and consequently,  $x$  is torsion too. Moreover, since  $N$  is subnormal in  $D^*$ , so is  $N'$ . Hence, by [13, Theorem 8],  $N' \subseteq F$ . Thus,  $N$  is solvable, and by [24, 14.4.4, p. 440],  $N \subseteq F$ .  $\square$

In Herstein's conjecture a subgroup  $N$  is required to be radical over center  $F$  of  $D$ . What happen if  $N$  is required to be radical over some proper division subring of  $D$  (which not necessarily coincides with  $F$ )? In the other words, the following question should be interesting: “Let  $D$  be a division ring and  $K$  be a proper division subring of  $D$  and given a subnormal subgroup  $N$  of  $D^*$ . If  $N$  is radical over  $K$ , then is it contained in center  $F$  of  $D$ ?” In the following we give the affirmative answer to this question for a weakly locally finite ring  $D$  and a normal subgroup  $N$ .

**Lemma 6.5.** *Let  $D$  be a weakly locally finite division ring with center  $F$  and  $N$  be a subnormal subgroup of  $D^*$ . If for every elements  $x, y \in N$ , there exists some positive integer  $n_{xy}$  such that  $x^{n_{xy}}y = yx^{n_{xy}}$ , then  $N \subseteq F$ .*

*Proof.* Since  $N$  is subnormal in  $D^*$ , there exists the following series of subgroups

$$N = N_1 \triangleleft N_2 \triangleleft \dots \triangleleft N_r = D^*.$$

Suppose that  $x, y \in N$ . Let  $K$  be the division subring of  $D$  generated by  $x$  and  $y$ . Then,  $K$  is centrally finite. By putting  $M_i = K \cap N_i, \forall i \in \{1, \dots, r\}$  we obtain the

following series of subgroups

$$M_1 \triangleleft M_2 \triangleleft \dots \triangleleft M_r = K^*.$$

For any  $a \in M_1 \leq N_1 = N$ , suppose that  $n_{ax}$  and  $n_{ay}$  are positive integers such that  $a^{n_{ax}}x = xa^{n_{ax}}$  and  $a^{n_{ay}}y = ya^{n_{ay}}$ . Then, for  $n := n_{ax}n_{ay}$  we have  $a^n = (a^{n_{ax}})^{n_{ay}} = (xa^{n_{ax}}x^{-1})^{n_{ay}} = xa^{n_{ax}n_{ay}}x^{-1} = xa^n x^{-1}$ , and  $a^n = (a^{n_{ay}})^{n_{ax}} = (ya^{n_{ay}}y^{-1})^{n_{ax}} = ya^{n_{ay}n_{ax}}y^{-1} = ya^n y^{-1}$ . Therefore  $a^n \in Z(K)$ . Hence  $M_1$  is radical over  $Z(K)$ . By Theorem 6.4,  $M_1 \subseteq Z(K)$ . In particular,  $x$  and  $y$  commute with each other. Consequently,  $N$  is abelian group. By [24, 14.4.4, p. 440],  $N \subseteq F$ .  $\square$

**Theorem 6.6.** *Let  $D$  be a weakly locally finite division ring with center  $F$  and  $K$  be a proper division subring of  $D$ . Then, every normal subgroup of  $D^*$  which is radical over  $K$  is contained in  $F$ .*

*Proof.* Assume that  $N$  is a normal subgroup of  $D^*$  which is radical over  $K$ , and  $N$  is not contained in the center  $F$ . If  $N \setminus K = \emptyset$ , then  $N \subseteq K$ . By [24, p. 433], either  $K \subseteq F$  or  $K = D$ . Since  $K \neq D$  by the assertion, it follows that  $K \subseteq F$ . Hence  $N \subseteq F$ , that contradicts to the assertion. Thus, we have  $N \setminus K \neq \emptyset$ .

Now, to complete the proof of our theorem we shall show that the elements of  $N$  satisfy the requirements of Lemma 6.5. Thus, suppose that  $a, b \in N$ . We examine the following cases:

*Case 1:  $a \in K$ .*

*Subcase 1.1:  $b \notin K$ .*

We shall prove that there exists some positive integer  $n$  such that  $a^n b = ba^n$ . Thus, suppose that  $a^n b \neq ba^n$  for any positive integer  $n$ . Then,  $a + b \neq 0$ ,  $a \neq \pm 1$  and  $b \neq \pm 1$ . So we have

$$x = (a + b)a(a + b)^{-1}, y = (b + 1)a(b + 1)^{-1} \in N.$$

Since  $N$  is radical over  $K$ , we can find some positive integers  $m_x$  and  $m_y$  such that

$$x^{m_x} = (a + b)a^{m_x}(a + b)^{-1}, y^{m_y} = (b + 1)a^{m_y}(b + 1)^{-1} \in K.$$

Putting  $m = m_x m_y$ , we have

$$x^m = (a + b)a^m(a + b)^{-1}, y^m = (b + 1)a^m(b + 1)^{-1} \in K.$$

Direct calculations give the equalities

$$x^m b - y^m b + x^m a - y^m = x^m(a + b) - y^m(b + 1) = (a + b)a^m - (b + 1)a^m = a^m(a - 1),$$

from that we get the following equality

$$(x^m - y^m)b = a^m(a - 1) + y^m - x^m a.$$

If  $(x^m - y^m) \neq 0$ , then  $b = (x^m - y^m)^{-1}[a^m(a - 1) + y^m - x^m a] \in K$ , that is a contradiction to the choice of  $b$ . Therefore  $(x^m - y^m) = 0$  and consequently,  $a^m(a - 1) = y^m(a - 1)$ . Since  $a \neq 1$ ,  $a^m = y^m = (b + 1)a^m(b + 1)^{-1}$  and it follows that  $a^m b = ba^m$ , a contradiction.

*Subcase 1.2:  $b \in K$ .*

Consider an element  $x \in N \setminus K$ . Since  $xb \notin K$ , by Subcase 1.1, there exist some positive integers  $r, s$  such that  $a^r x b = x b a^r$  and  $a^s x = x a^s$ . From these equalities it follows that  $a^{rs} = (xb)^{-1} a^{rs} (xb) = b^{-1} (x^{-1} a^{rs} x) b = b^{-1} a^{rs} b$ , and consequently,  $a^{rs} b = b a^{rs}$ .

*Case 2:  $a \notin K$ .*

Since  $N$  is radical over  $K$ , there exists some positive integer  $m$  such that  $a^m \in K$ . By Case 1, there exists some positive integer  $n$  such that  $a^{mn}b = ba^{mn}$ .  $\square$

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